

# KATO SMOOTHING AND STRICHARTZ ESTIMATES FOR WAVE EQUATIONS WITH MAGNETIC POTENTIALS

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**ABSTRACT.** Let  $H$  be a selfadjoint operator and  $A$  a closed operator on a Hilbert space  $\mathcal{H}$ . If  $A$  is  $H$ -(super)smooth in the sense of Kato-Yajima, we prove that  $AH^{-\frac{1}{4}}$  is  $\sqrt{H}$ -(super)smooth. This allows to include wave and Klein-Gordon equations in the abstract theory at the same level of generality as Schrödinger equations.

We give a few applications and in particular, based on the resolvent estimates of Erdogan, Goldberg and Schlag [9], we prove Strichartz estimates for wave equations perturbed with large magnetic potentials on  $\mathbb{R}^n$ ,  $n \geq 3$ .

## 1. INTRODUCTION

In his fundamental 1965 paper [16], Kato develops a theory of similarity for small perturbations  $H(\epsilon) = H + \epsilon V$  of an unbounded operator  $H$  on a Hilbert space  $\mathcal{H}$ , by constructing a bounded wave operator  $W(\epsilon)$  with the property  $H(\epsilon) = W(\epsilon)HW^{-1}(\epsilon)$ . In the selfadjoint case  $H = H^*$  the theory can be precised and provides a bridge between the dispersive properties of the Schrödinger flow  $e^{itH}$  and uniform estimates for the resolvent operator  $R(z) = (H - z)^{-1}$ .

The relevance of Kato's theory in the study of dispersive equations was understood already in [25] and [15]. A remarkable application was given in [21] where Kato smoothing was used to give a simple proof of Strichartz estimates for the flow  $e^{it(-\Delta+V)}$  perturbed by a short range potential  $|V(x)| \lesssim \langle x \rangle^{-2-\epsilon}$ . The corresponding result for short range magnetic potentials was proved in [5] in the case of small potentials and [9] for large potentials (for recent related results, see also [4], [18], [8], [7], [11]).

It is natural to investigate applications of Kato's theory to the corresponding wave-Klein-Gordon flow  $e^{it\sqrt{H+\nu}}$  (with  $H + \nu \geq 0$ ). The standard approach is a reduction to the Schrödinger flow  $e^{itK}$  where

$$K = \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix} \quad \Rightarrow \quad \exp(itK) = \begin{pmatrix} \cos(t\sqrt{H}) & \frac{i}{\sqrt{H}} \sin(t\sqrt{H}) \\ i\sqrt{H} \sin(t\sqrt{H}) & \cos(t\sqrt{H}) \end{pmatrix}$$

however this path leads to a loss in the sharpness of the estimates, and in some cases it requires some ad-hoc argument to prove the necessary resolvent estimates for  $K$  (see e.g. [19], [6] or [2]).

The first goal of this note is to deduce smoothing estimates for abstract wave equations within the framework of Kato's theory: given a non negative selfadjoint operator  $H$  and a closed operator  $A$  on the Hilbert space  $\mathcal{H}$ , we prove that

$$A \text{ is } H\text{-(super)smooth} \implies AH^{-\frac{1}{4}} \text{ is } \sqrt{H}\text{-(super)smooth}$$

in the sense of Kato-Yajima; see Section 2 for definitions and details and in particular Theorem 2.4. Actually we prove that if  $H + \nu \geq 0$  for some  $\nu \in \mathbb{R}$  and  $A$  is  $H$ -(super)smooth, then  $A(H + \nu)^{-\frac{1}{4}}$  is  $\sqrt{H + \nu}$ -(super)smooth.

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It is clear that the range of applications is quite wide. In Section 3 we picked three. The first two are mostly known results: smoothing estimates for the flows generated by powers of the Laplacian; and the smoothing estimates for wave equations with potentials of critical decay which were obtained in [2].

As a main application of the abstract result, we prove sharp Strichartz estimates for wave equations perturbed with large magnetic potentials, thus extending the result for small potentials in [5]. This result is based on the resolvent estimate due to Erdogan, Goldberg and Schlag [9] for the magnetic Schrödinger operator

$$H = (i\nabla + A(x))^2 + V(x) \quad \text{on } \mathbb{R}^n, n \geq 3$$

under the following assumptions:  $A(x) \in \mathbb{R}^n$ ,  $V(x) \in \mathbb{R}$ ; moreover for some  $C, \epsilon_0 > 0$

$$|A(x)| + \langle x \rangle |V(x)| \leq C \langle x \rangle^{-1-\epsilon_0},$$

$$\forall 0 < \epsilon < \epsilon_0, \quad \langle x \rangle^{1+\epsilon} A(x) \in \dot{H}_{2n}^{\frac{1}{2}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$$

( $\|u\|_{\dot{H}_q^s} := \| |D|^s u \|_{L^q}$ ,  $|D|^s := (-\Delta)^{\frac{s}{2}}$ ); finally, 0 is not an eigenvalue of  $H$  nor a resonance, in the sense that

$$\langle x \rangle^{\frac{n-4}{2}-} u(x) \in L^2(\mathbb{R}^n), \quad Hu = 0 \implies u \equiv 0.$$

In dimension  $n \geq 5$  it is sufficient to assume that 0 is not an eigenvalue. Under these assumptions, Erdogan, Goldberg and Schlag proved that the Schrödinger flow  $e^{itH}$  satisfies the same Strichartz estimates as the free flow  $e^{it\Delta}$ . In Section 3.3 we combine their resolvent estimate with Theorem 3.2 to prove:

**Theorem 1.1.** *Assume  $H$  is selfadjoint, nonnegative and satisfies the previous assumptions. Then the wave flow satisfies the non-endpoint Strichartz estimates*

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} e^{it\sqrt{H}} f \|_{L^p L^q} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}},$$

and

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} \sin(t\sqrt{H}) H^{-\frac{1}{2}} g \|_{L^p L^q} \lesssim \|g\|_{\dot{H}^{-\frac{1}{2}}},$$

for all  $2 < p \leq \infty$ ,  $2 \leq q < \frac{2(n-1)}{(n-3)}$  with  $2p^{-1} + (n-1)q^{-1} = (n-1)2^{-1}$ .

*Remark 1.1.* It is possible to prove the estimates also at the endpoint, using the ideas in [14], but this would lead us too far from the main goal of the paper.

*Remark 1.2.* By the same methods one can prove non-endpoint Strichartz estimates for the Klein-Gordon flow  $e^{it\sqrt{H+\nu}}$  perturbed with large potentials, provided  $H + \nu \geq 0$ . We omit the details.

## 2. ABSTRACT KATO SMOOTHING

In this section we review the basics of Kato's theory for the Schrödinger equation as developed in [16] and [17] (see also [20] and [19]), adding concise proofs when necessary, and then we extend it to wave type equations.

Throughout this section  $\mathcal{H}, \mathcal{H}_1$  are Hilbert spaces and  $H$  is a selfadjoint operator on  $\mathcal{H}$  with domain  $D(H)$ . For  $z \in \mathbb{C} \setminus \mathbb{R}$ , let  $R(z) = (H - z)^{-1}$  be the resolvent operator of  $H$ , and

$$\Im R(z) = (2i)^{-1}(R(z) - R(\bar{z}))$$

its imaginary part. A *step function*  $u(t) : \mathbb{R} \rightarrow \mathcal{H}$  is a measurable function of bounded support taking a finite number of values; measurability and integrals of Hilbert-valued functions are in the sense of Bochner. We also write for short  $L^p \mathcal{H} = L^p(\mathbb{R}; \mathcal{H})$ . The following terminology was introduced in [17]:

**Definition 2.1.** A closed operator  $A$  from  $\mathcal{H}$  to  $\mathcal{H}_1$  with dense domain  $D(A)$  is called:

(i) *H-smooth*, with constant  $a$ , if  $\exists \epsilon_0$  such that for every  $\epsilon, \lambda \in \mathbb{R}$  with  $0 < |\epsilon| < \epsilon_0$  the following uniform bound holds:

$$|(\Im R(\lambda + i\epsilon)A^*v, A^*v)_{\mathcal{H}_1}| \leq a\|v\|_{\mathcal{H}_1}^2, \quad v \in D(A^*); \quad (2.1)$$

(ii) *H-supersmooth*, with constant  $a$ , if in place of (2.1) one has

$$|(R(\lambda + i\epsilon)A^*v, A^*v)_{\mathcal{H}_1}| \leq a\|v\|_{\mathcal{H}_1}^2, \quad v \in D(A^*). \quad (2.2)$$

*Remark 2.1.* Note that (2.1) implies that the range of  $\Im R(z)A^*$  is contained in  $D(A)$  and the (selfadjoint nonnegative) operator  $A\Im R(z)A^*$  is a bounded operator on  $\mathcal{H}_1$  with norm equal to  $a$ . In a similar way, (2.2) implies that the range of  $R(z)A^*$  is contained in  $D(A)$  and the operator  $AR(z)A^*$  is a bounded operator on  $\mathcal{H}_1$  with norm not exceeding  $2a$  (and not smaller than  $a$ ).

**Theorem 2.2.** Let  $A : \mathcal{H} \rightarrow \mathcal{H}_1$  be a closed operator with dense domain  $D(A)$ . Then  $A$  is *H-smooth* with constant  $a$  if and only if, for any  $v \in \mathcal{H}$ , one has  $e^{-itH}v \in D(A)$  for almost every  $t$  and the following estimate holds:

$$\|Ae^{-itH}v\|_{L^2\mathcal{H}_1} \leq 2a^{\frac{1}{2}}\|v\|_{\mathcal{H}}. \quad (2.3)$$

*Proof.* This is proved in Lemma 3.6 and Theorem 5.1 of [16] (see also Theorem XIII.25 in [20]).  $\square$

Thus *H-smoothness* is equivalent to the smoothing estimate (2.3) for the homogeneous flow  $e^{-itH}$ . By similar methods, it is not difficult to see that *H-supersmoothness* is equivalent to a *nonhomogeneous* estimate (compare also with [19]):

**Theorem 2.3.** Let  $A : \mathcal{H} \rightarrow \mathcal{H}_1$  be a closed operator with dense domain  $D(A)$ . Assume  $A$  is *H-supersmooth* with constant  $a$ . Then  $e^{-itH}v \in D(A)$  for almost any  $t \in \mathbb{R}$  and any  $v \in \mathcal{H}$ ; moreover, for any step function  $h(t) : \mathbb{R} \rightarrow D(A^*)$ ,  $Ae^{-i(t-s)H}A^*h(s)$  is Bochner integrable in  $s$  over  $[0, t]$  (or  $[t, 0]$ ) and satisfies, for all  $\epsilon \in (-\epsilon_0, \epsilon_0)$ , the estimate

$$\|e^{-\epsilon t} \int_0^t Ae^{-i(t-s)H}A^*h(s)ds\|_{L^2\mathcal{H}_1} \leq 2a\|e^{-\epsilon t}h(t)\|_{L^2\mathcal{H}_1}. \quad (2.4)$$

Conversely, if (2.4) holds, then  $A$  is *H-supersmooth* with constant  $2a$ .

*Proof.* Assume  $A$  is *H-supersmooth*, hence in particular *H-smooth*. Then from the previous Theorem we know that  $e^{-itH}v \in D(A)$  for a.e.  $t$ . Denote for a function  $v(t) : \mathbb{R} \rightarrow \mathcal{H}$  its Laplace transforms by

$$\tilde{v}_+(z) = \int_0^{+\infty} e^{izt}v(t)dt, \quad \Im z > 0 \quad \text{and} \quad \tilde{v}_-(z) = \int_{-\infty}^0 e^{izt}v(t)dt, \quad \Im z < 0.$$

Note that both integrals converge if  $\|v(t)\|_{\mathcal{H}}$  grows at most polynomially. In particular, if  $F(t) \in L^\infty(\mathbb{R}; \mathcal{H})$  and

$$v(t) = \int_0^t e^{-i(t-s)H}F(s)ds, \quad (2.5)$$

we have the well known identities

$$\tilde{v}_\pm(z) = i^{-1}R(z)\tilde{F}(z), \quad \pm \Im z > 0. \quad (2.6)$$

Now let  $h(t) : \mathbb{R} \rightarrow D(A^*)$  be a step function, define

$$v(t) = \int_0^t e^{-i(t-s)H}A^*h(s)ds$$

and consider the Laplace transforms of  $v(t)$ ; we have by (2.6)

$$\tilde{v}_\pm(z) = i^{-1}R(z)\widetilde{A^*h}(z) = i^{-1}R(z)A^*\tilde{h}(z), \quad \pm \Im z > 0$$

where  $\widetilde{A^*h} = A^*\tilde{h}$  follows by Hille's theorem (Theorem 3.7.12 in [12]).

Note also that  $v(t) \in D(A)$  for all  $t$ . To see this, write explicitly

$$h(t) = \sum_{j=1}^N \mathbf{1}_{E_j}(t) h_j$$

for some  $h_j \in D(A^*)$  and some measurable disjoint bounded sets  $E_j \subset \mathbb{R}$  ( $\mathbf{1}_E$  denotes the characteristic function of  $E$ ). Then we have, for fixed  $t$ ,

$$v(t) = \sum \int_0^t \mathbf{1}_{E_j}(s) e^{isH} v_j ds, \quad v_j = e^{-itH} h_j$$

and we know that  $Ae^{isH} v_j \in L^2(0, t; \mathcal{H}_1) \subset L^1(0, t; \mathcal{H}_1)$  by the previous Theorem. Thus by Hille's theorem we deduce that  $v(t) \in D(A)$  and

$$Av(t) = \int_0^t Ae^{-i(t-s)H} A^* h(s) ds. \quad (2.7)$$

Now take a second step function  $g(t) : \mathbb{R} \rightarrow D(A^*)$  and apply Parseval's identity: for all  $\epsilon > 0$ ,

$$2\pi \int_{-\infty}^{+\infty} (\tilde{v}_{\pm}(\lambda \pm i\epsilon), \widetilde{A^*g}(\lambda \pm i\epsilon))_{\mathcal{H}} d\lambda = \pm \int_0^{\pm\infty} e^{-2\epsilon|t|} (v(t), A^*g(t))_{\mathcal{H}} dt. \quad (2.8)$$

Using again Hille's theorem to prove  $\widetilde{A^*g} = A^*\tilde{g}$ , and the supersmoothness assumption (see also Remark 2.1) we can write

$$|(\tilde{v}_{\pm}(z), \widetilde{A^*g}(z))_{\mathcal{H}}| = |(AR(z)A^*\tilde{h}(z), \tilde{g}(z))_{\mathcal{H}_1}| \leq 2a \|\tilde{h}(z)\|_{\mathcal{H}_1} \|\tilde{g}(z)\|_{\mathcal{H}_1}$$

and plugging into (2.8) we obtain with a last application of Parseval

$$\begin{aligned} |\int_0^{\pm\infty} e^{-2\epsilon|t|} (v(t), A^*g(t))_{\mathcal{H}} dt| &\leq 4\pi a (\int_{-\infty}^{+\infty} \|\tilde{h}(\lambda \pm i\epsilon)\|_{\mathcal{H}_1}^2 d\lambda)^{\frac{1}{2}} (\int_{-\infty}^{+\infty} \|\tilde{g}(\lambda \pm i\epsilon)\|_{\mathcal{H}_1}^2 d\lambda)^{\frac{1}{2}} \\ &= 2a \|e^{-\epsilon|t|} h\|_{L^2(\mathbb{R}^{\pm}; \mathcal{H}_1)} \|e^{-\epsilon|t|} g\|_{L^2(\mathbb{R}^{\pm}; \mathcal{H}_1)}. \end{aligned}$$

Recalling that  $v(t) \in D(A)$  we arrive at

$$|\int_0^{\pm\infty} (e^{-\epsilon t} Av(t), e^{-\epsilon t} g(t))_{\mathcal{H}_1} dt| \leq 2a \|e^{-\epsilon t} h\|_{L^2(\mathbb{R}^{\pm}; \mathcal{H}_1)} \|e^{-\epsilon t} g\|_{L^2(\mathbb{R}^{\pm}; \mathcal{H}_1)}.$$

By density of step functions (and of  $D(A^*)$ ) this implies

$$\|e^{-\epsilon t} Av\|_{L^2 \mathcal{H}_1} \leq 2a \|e^{-\epsilon t} h\|_{L^2 \mathcal{H}_1}$$

and recalling (2.7) we obtain (2.4) (including the case  $\epsilon = 0$  which is obtained by taking the limit  $\epsilon \rightarrow 0$ ).

We now prove the converse statement. Assume (2.4) holds for any step function  $h : \mathbb{R}^+ \rightarrow D(A^*)$ ; then by a simple approximation argument one sees that (2.4) holds for any function  $h(t)$  of the form

$$h(t) = \sigma(t) h_0, \quad h_0 \in D(A^*), \quad \sigma(t) \in L^2(\mathbb{R}^+);$$

thus, writing

$$w(t) = \int_0^t Ae^{-i(t-s)H} A^* h(s) ds$$

and applying (2.4) we get

$$\|e^{-\epsilon t} w(t)\|_{L^2(\mathbb{R}^+; \mathcal{H}_1)} \leq 2a \|e^{-\epsilon t} h\|_{L^2 \mathcal{H}_1}.$$

By Parseval we have then

$$(2\pi)^{\frac{1}{2}} \|\tilde{w}_+(\lambda + i\epsilon)\|_{L_{\lambda}^2 \mathcal{H}_1} = \|e^{-\epsilon t} w(t)\|_{L^2(\mathbb{R}^+; \mathcal{H}_1)} \leq 2a \|e^{-\epsilon t} h\|_{L^2 \mathcal{H}_1} = (2\pi)^{\frac{1}{2}} 2a \|\tilde{h}(\lambda + i\epsilon)\|_{L_{\lambda}^2 \mathcal{H}_1}$$

so that

$$\|\tilde{w}_+(\lambda + i\epsilon)\|_{L_{\lambda}^2 \mathcal{H}_1} \leq 2a \|\tilde{h}(\lambda + i\epsilon)\|_{L_{\lambda}^2 \mathcal{H}_1}. \quad (2.9)$$

Recalling (2.6), we have

$$\tilde{w}_+(\lambda + i\epsilon) = i^{-1} AR(\lambda + i\epsilon) A^* \tilde{h}_+(\lambda + i\epsilon)$$

where

$$\tilde{h}_+(\lambda + i\epsilon) = \int_0^{\infty} e^{i(\lambda + i\epsilon)t} \sigma(t) dt \cdot h_0 = \tilde{\sigma}_+(\lambda + i\epsilon) \cdot h_0$$

thus we can write

$$\|\tilde{w}_+(\lambda + i\epsilon)\|_{L_{\lambda}^2 \mathcal{H}_1}^2 = \int_{-\infty}^{\infty} \phi(\lambda + i\epsilon) |\tilde{\sigma}(\lambda + i\epsilon)|^2 d\lambda, \quad \phi(\lambda) = \|AR(\lambda + i\epsilon) A^* h_0\|_{\mathcal{H}_1}^2$$

Plugging into (2.9) we obtain

$$\int_{-\infty}^{\infty} \phi(\lambda + i\epsilon) |\tilde{\sigma}(\lambda + i\epsilon)|^2 d\lambda \leq (2a)^2 \|h_0\|_{\mathcal{H}_1}^2 \int_{-\infty}^{\infty} |\tilde{\sigma}(\lambda + i\epsilon)|^2 d\lambda$$

and recalling that  $\sigma \in L^2$  is arbitrary, we deduce

$$\sup_{\lambda \in \mathbb{R}} |\phi(\lambda + i\epsilon)| \leq 2a \|h_0\|_{\mathcal{H}_1}$$

which is precisely the  $H$ -supersmoothness condition for  $A$ .  $\square$

*Remark 2.2.* It would be possible to prove a more general result where  $h$  is taken to be a generic function in  $L^2\mathcal{H}_1$  with values in  $D(A^*)$ , instead of a step function. However this makes the proof of (2.7) rather involved (in particular, the integral in (2.7) must be interpreted in the sense of Pettis). Since in concrete applications the final approximation step becomes trivial, we opted for a simpler statement expressed in terms of step functions.

We now show that the smoothness property is inherited by the square root of  $H$ :

**Theorem 2.4.** *Let  $\nu \in \mathbb{R}$  with  $H + \nu \geq 0$  and  $H + \nu$  injective. Assume  $A$  and  $A(H + \nu)^{-\frac{1}{4}}$  are closed operators with dense domain from  $\mathcal{H}$  to  $\mathcal{H}_1$ .*

*(i) If  $A$  is  $H$ -smooth with constant  $a$ , then  $A(H + \nu)^{-\frac{1}{4}}$  is  $\sqrt{H + \nu}$ -smooth with constant  $C = (\pi + 3)a$ . In particular, we have the estimate*

$$\|Ae^{-it\sqrt{H+\nu}}v\|_{L^2\mathcal{H}_1} \leq 2C^{\frac{1}{2}} \|(H + \nu)^{\frac{1}{4}}v\|_{\mathcal{H}}, \quad \forall v \in D((H + \nu)^{\frac{1}{4}}). \quad (2.10)$$

*(ii) If  $A$  is  $H$ -supersmooth with constant  $a$ , then  $A(H + \nu)^{-\frac{1}{4}}$  is  $\sqrt{H + \nu}$ -supersmooth with constant  $C = (\pi + 3)a$ . In particular, we have the estimate*

$$\left\| \int_0^t Ae^{-i(t-s)\sqrt{H+\nu}}(H + \nu)^{-\frac{1}{2}}A^*h(s)ds \right\|_{L^2\mathcal{H}_1} \leq 2(\pi + 3)a\|h\|_{L^2\mathcal{H}_1} \quad (2.11)$$

for any step function  $h : \mathbb{R} \rightarrow D((H + \nu)^{-\frac{1}{4}}A^*)$ .

*Proof.* We give a detailed proof for case (ii) and at the end we shall list the (minor) modifications needed to prove (i). Note that by renaming the operator  $H$ , it is not restrictive to assume  $\nu = 0$ .

We need to prove a uniform bound in  $\epsilon_0 > \Im z > 0$  for the operators

$$AH^{-\frac{1}{4}}(\sqrt{H} - z)^{-1}H^{-\frac{1}{4}}A^* : \mathcal{H}_1 \rightarrow \mathcal{H}_1. \quad (2.12)$$

We have (the notation  $S \subset T$  means that the operator  $T$  extends  $S$ )

$$H^{-\frac{1}{4}}(\sqrt{H} - z)^{-1}H^{-\frac{1}{4}} \subset H^{-\frac{1}{2}}(\sqrt{H} + z)(H - z^2)^{-1} = (I + zH^{-\frac{1}{2}})R(z^2).$$

By assumption we already know that  $AR(z^2)A^*$  is uniformly bounded with norm  $\leq a$ , thus it remains to prove that

$$AzH^{-\frac{1}{2}}R(z^2)A^* : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \quad (2.13)$$

is uniformly bounded. We plan to estimate this operator using the spectral theorem. The obstruction to such an estimate is due to a singularity at  $\lambda = z^2$  in the spectral representation (see below); however, we can remove this singularity by adding (or subtracting) a suitable operator for which we have already an estimate. Indeed, using the resolvent identity

$$R(z^2) - R(-|z|^2) = (z^2 + |z|^2)R(z^2)R(|z|^2)$$

we see, again by assumption, that the operator

$$A(z^2 + |z|^2)R(z^2)R(|z|^2)A^*$$

is also bounded with norm  $\leq 2a$ . Adding or subtracting this from (2.13), we see that it is sufficient to prove that at least one of the operators

$$Q_{\pm}(z) = A \left[ zH^{-\frac{1}{2}} \pm (z^2 + |z|^2)R(|z|^2) \right] R(z^2)A^*$$

is bounded uniformly in  $z$ . By Stone's formula we can write

$$Q_{\pm}(z)v = \lim_{\epsilon \downarrow 0} \int_0^{\infty} \frac{1}{\lambda - z^2} \left( \frac{z}{\sqrt{\lambda}} \pm \frac{z^2 + |z|^2}{\lambda + |z|^2} \right) A \Im R(\lambda + i\epsilon) A^* v d\lambda$$

and using one last time the assumption on the uniform bound  $\leq a$  for the norm of the operator  $A \Im R(\lambda + i\epsilon) A^*$ , we obtain

$$\|Q_{\pm}(z)v\|_{\mathcal{H}_1} \leq \int_0^{\infty} \frac{1}{|\lambda - z^2|} \left| \frac{z}{\sqrt{\lambda}} \pm \frac{z^2 + |z|^2}{\lambda + |z|^2} \right| d\lambda \cdot a \|v\|_{\mathcal{H}_1}.$$

We now check that the last integral is uniformly bounded for  $\Im z > 0$ . Let  $z = e^{i\theta}|z|$ , with  $|z| > 0$  and  $0 < \theta < \pi$ , then by rescaling  $\lambda = |z|^2 \mu$  we have

$$I_{\pm} \equiv \int_0^{\infty} \frac{1}{|\lambda - z^2|} \left| \frac{z}{\sqrt{\lambda}} \pm \frac{z^2 + |z|^2}{\lambda + |z|^2} \right| d\lambda = \int_0^{\infty} \frac{1}{|\mu - e^{2i\theta}|} \left| \frac{e^{i\theta}}{\sqrt{\mu}} \pm \frac{e^{2i\theta} + 1}{\mu + 1} \right| d\mu.$$

In the case  $0 < \theta \leq \frac{\pi}{2}$ , we prove that  $I_-$  is bounded. Indeed, using the identity

$$|e^{i\theta}(\mu + 1) - \sqrt{\mu}(e^{2i\theta} + 1)| = |\sqrt{\mu}e^{i\theta} - 1| \cdot |\sqrt{\mu} - e^{i\theta}| = |\sqrt{\mu} - e^{i\theta}|^2$$

we obtain

$$I_- = \int_0^{\infty} \frac{1}{|\mu - e^{2i\theta}|} \frac{|\sqrt{\mu} - e^{i\theta}|^2}{\sqrt{\mu}(\mu + 1)} d\mu.$$

We then simplify the first factor in  $|\mu - e^{2i\theta}| = |\sqrt{\mu} - e^{i\theta}| \cdot |\sqrt{\mu} + e^{i\theta}|$  to obtain

$$I_- = \int_0^{\infty} \frac{|\sqrt{\mu} - e^{i\theta}|}{\sqrt{\mu}(\mu + 1) \cdot |\sqrt{\mu} + e^{i\theta}|} d\mu \leq \int_0^{\infty} \frac{1}{\sqrt{\mu}(\mu + 1)} d\mu = \pi.$$

Thus if  $0 < \theta \leq \frac{\pi}{2}$  we have  $\|Q_-(z)v\|_{\mathcal{H}_1} \leq \pi a \|v\|_{\mathcal{H}_1}$ ; this implies a bound  $(\pi + 2)a$  for the norm of the operator (2.13) for such  $z$ , and a bound  $(\pi + 3)a$  for the norm of (2.12).

On the other hand, in the case  $\frac{\pi}{2} < \theta < \pi$  we prove that  $I_+$  is bounded: we have

$$I_+ = \int_0^{\infty} \frac{|(\mu + 1)e^{i\theta} + \sqrt{\mu}(e^{2i\theta} + 1)|}{\sqrt{\mu}(\mu + 1)|\mu - e^{2i\theta}|} d\mu$$

and using the identity

$$|(\mu + 1)e^{i\theta} + \sqrt{\mu}(e^{2i\theta} + 1)| = |\sqrt{\mu} + e^{i\theta}| \cdot |\sqrt{\mu}e^{i\theta} + 1| = |\sqrt{\mu} + e^{i\theta}|^2$$

and simplifying the first factor in

$$|\mu - e^{2i\theta}| = |\sqrt{\mu} + e^{i\theta}| \cdot |\sqrt{\mu} - e^{i\theta}|$$

we obtain

$$I_+ = \int_0^{\infty} \frac{|\sqrt{\mu} + e^{i\theta}|}{\sqrt{\mu}(\mu + 1)|\sqrt{\mu} - e^{i\theta}|} d\mu \leq \int_0^{\infty} \frac{1}{\sqrt{\mu}(\mu + 1)} d\mu = \pi.$$

Thus we obtain the same bound as before for the remaining values of  $z$ , proving that (2.12) is  $\sqrt{H}$ -supersmooth with a constant  $(\pi + 3)a$ . The final estimate (2.11) is a direct consequence of Theorem 2.3.

It is easy to modify the previous argument for the proof of (i): indeed, the bound for

$$AH^{-\frac{1}{4}} \Im(\sqrt{H} - z)^{-1} H^{-\frac{1}{4}} A^*$$

is reduced as before to bounds for the operators

$$\tilde{Q}_{\pm}(z) = A \Im \left\{ \left[ zH^{-\frac{1}{2}} \pm (z^2 + |z|^2)R(|z|^2) \right] R(z^2) \right\} A^*$$

which follow exactly from the computations above. Finally applying Theorem 2.2 we obtain the estimate

$$\|Ae^{-it\sqrt{H+\nu}}(H + \nu)^{-\frac{1}{4}}v\|_{L^2\mathcal{H}_1} \leq 2C^{\frac{1}{2}}\|v\|_{\mathcal{H}},$$

whence we obtain (2.10).  $\square$

It is not difficult to see that the injectivity assumption on  $H + \nu$  is not necessary, in the following sense. If  $\ker(H + \nu) \neq \{0\}$ , denoting by  $P$  be the orthogonal projection onto  $\mathcal{K} = \ker(H + \nu)^\perp$  and by  $(H + \nu)^{-\frac{1}{4}}$  the operator  $(H + \nu)|_{\mathcal{K}}^{-\frac{1}{4}}$ , we see that the operator  $(H + \nu)^{-\frac{1}{4}}P$  is closed and densely defined on  $\mathcal{H}$ . Moreover, if  $A$  is  $H$ -smooth and  $v \in \ker(H + \nu)$ , from the smoothing estimate (2.3) it follows immediately that  $Av = 0$ . Thus we obtain the following more general result:

**Corollary 2.5.** *Let  $\nu \in \mathbb{R}$  with  $H + \nu \geq 0$  and let  $P$  be the orthogonal projection onto  $\ker(H + \nu)^\perp$ . Assume  $A$  and  $A(H + \nu)^{-\frac{1}{4}}P$  are closed operators with dense domain from  $\mathcal{H}$  to  $\mathcal{H}_1$ .*

(i) *If  $A$  is  $H$ -smooth with constant  $a$ , then  $A(H + \nu)^{-\frac{1}{4}}P$  is  $\sqrt{H + \nu}$ -smooth with constant  $C = (\pi + 3)a$ . In particular, we have the estimate*

$$\|Ae^{-it\sqrt{H+\nu}}v\|_{L^2\mathcal{H}_1} \leq 2C^{\frac{1}{2}}\|(H + \nu)^{\frac{1}{4}}v\|_{\mathcal{H}}, \quad \forall v \in D((H + \nu)^{\frac{1}{4}}). \quad (2.14)$$

(ii) *If  $A$  is  $H$ -supersmooth with constant  $a$ , then  $A(H + \nu)^{-\frac{1}{4}}P$  is  $\sqrt{H + \nu}$ -supersmooth with constant  $C = (\pi + 3)a$ . In particular, we have the estimate*

$$\left\| \int_0^t Ae^{-i(t-s)\sqrt{H+\nu}}(H + \nu)^{-\frac{1}{2}}PA^*h(s)ds \right\|_{L^2\mathcal{H}_1} \leq C\|h\|_{L^2\mathcal{H}_1} \quad (2.15)$$

for any step function  $h : \mathbb{R} \rightarrow D((H + \nu)^{-\frac{1}{4}}PA^*)$ .

*Proof.* The proof is obtained simply by restricting to the closed subspace  $\ker(H + \nu)^\perp$  and applying the previous Theorem. Note that in case (i) we get the estimate

$$\|Ae^{-it\sqrt{H+\nu}}(H + \nu)^{-\frac{1}{4}}v\|_{L^2\mathcal{H}_1} \leq 2C^{\frac{1}{2}}\|v\|_{\mathcal{H}}, \quad \forall v \in \ker(H + \nu)^\perp$$

which gives (2.10) for  $v \in \ker(H + \nu)^\perp$ , while for  $v \in \ker(H + \nu)$  estimate (2.10) is trivial since the left hand side is identically 0 by the remark preceding the Theorem.  $\square$

*Remark 2.3.* By the same technique it is possible to prove smoothing estimates for the flows of the form  $e^{it(H+\nu)^{1/m}}$ ,  $m \geq 0$  integer; this might be interesting especially for  $m = 4$  since the equation  $i u_t + |D|^{\frac{1}{2}}u = 0$  is relevant in connection with water waves. We omit the details.

### 3. APPLICATIONS

The results of the previous Section are rather general and have a wide range of applications; here we shall mention just a few. We use the notations

$$|D|^\alpha = (-\Delta)^{\frac{\alpha}{2}}, \quad \langle D \rangle^\alpha = (1 - \Delta)^{\frac{\alpha}{2}}, \quad \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

The operators appearing in the following are intended to be closed extension of the corresponding operators defined on  $C_c^\infty(\mathbb{R}^n)$ .

**3.1. Powers of the Laplacian.** We begin by recalling a few smoothing estimates for operators with constant coefficients. Most of these results are well known or can be obtained directly via Fourier analysis; we review them both for the purpose of illustration and for later use below.

In the Kato-Yajima paper [17], the case  $H = -\Delta$  on  $\mathbb{R}^n$ ,  $n \geq 3$  is considered in detail, and in particular it is proved that

- $|x|^{\alpha-1}|D|^\alpha$  for  $0 \leq \alpha < 1/2$  is  $(-\Delta)$ -supersmooth
- $\langle x \rangle^{-1}\langle D \rangle^{\frac{1}{2}}$  is  $(-\Delta)$ -supersmooth
- the multiplication operator by any  $a(x) \in L^n(\mathbb{R}^n)$  is  $(-\Delta)$ -supersmooth

This result is precised in [24] as follows: for  $n \geq 2$ ,

- $|x|^{-\beta}|D|^{\alpha-\beta}$  is  $|D|^{2\alpha}$ -supersmooth for  $2\alpha > 1$ ,  $\beta \leq \alpha$  and  $\frac{1}{2} < \beta < \frac{n}{2}$ .

If instead of the supersmoothing property we restrict to the weaker smoothing property, several additional results are available ([23], [22] among the others). We mention in particular the following one, which was proved in [1], [3] and will be used below:

- $\langle x \rangle^{-\frac{1}{2}-\epsilon} |D|^{\frac{1}{2}}$  is  $(-\Delta)$ -smooth for  $n \geq 2$ ,  $\epsilon > 0$ .

The previous results, combined with Theorems 2.2, 2.3, 2.4, give immediately several estimates for time dependent flows. By Watanabe's result and Theorem 2.2 we get

$$\| |x|^{-\beta} |D|^{\alpha-\beta} e^{-it|D|^{2\alpha}} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2}, \quad 2\alpha > 1, \quad \beta \leq \alpha, \quad \frac{1}{2} < \beta < \frac{n}{2}.$$

This estimate is implied by the result in [10] (see also [13])

$$\| |x|^{-\beta} |D|^{\alpha-\beta} e^{-it|D|^{2\alpha}} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|\Lambda^{\frac{1}{2}-\beta} f\|_{L^2}, \quad \Lambda = (1 - \Delta_{\mathbb{S}^{n-1}})^{\frac{1}{2}}$$

valid for any  $\alpha > 0$  and  $\beta \in (\frac{1}{2}, \frac{n}{2})$ , with a further gain in angular regularity (note that  $\frac{1}{2} - \beta < 0$ ). However, our theory covers also the nonhomogeneous case. Indeed, applying Theorem 2.3 we get, for the same range of parameters as in Watanabe's result, the nonhomogeneous estimate

$$\| |x|^{-\beta} |D|^{\alpha-\beta} \int_0^t e^{-i(t-s)|D|^{2\alpha}} F(s) ds \|_{L^2(\mathbb{R}^{n+1})} \lesssim \| |x|^\beta |D|^{-\alpha+\beta} F \|_{L^2(\mathbb{R}^{n+1})}. \quad (3.1)$$

Estimate (3.1) does not include the case of the wave flow since  $\alpha > 1/2$  in [24]; however using Theorem 2.4 we obtain that (3.1) holds for all  $\alpha > 1/4$ .

Several other applications to constant coefficient equations are possible. As a final example we consider Klein-Gordon equations, which are covered by Theorem 2.3 with  $\mu = 1$ : we obtain the smoothing estimate

$$\| |x|^{-\beta} |D|^{1-\beta} e^{it\langle D \rangle} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|\langle D \rangle^{\frac{1}{2}} f\|_{L^2}. \quad (3.2)$$

**3.2. Potentials of critical decay.** Our second example is a simplification of a proof in [2], where Strichartz estimates were obtained for Schrödinger and wave equations of the form

$$iu_t - \Delta u + V(x)u = 0, \quad u_{tt} - \Delta u + V(x)u = 0.$$

$V(x)$  is a real valued potential satisfying the following assumptions: there exist  $C > 0$  and  $c < \frac{(n-2)^2}{4}$ ,  $n \geq 3$ , such that

$$\frac{C}{|x|^2} \geq V(x) \geq -\frac{c}{|x|^2} \quad \text{and} \quad -\partial_r(|x|V(x)) \geq -\frac{c}{|x|^2}, \quad \partial_r := \frac{x}{|x|} \cdot \nabla_x \quad (3.3)$$

(the actual assumptions are slightly more general). The crucial step in [2] is Theorem 3, claiming that

$$|x|^{-1} \text{ is } (-\Delta + V)\text{-supersmoothing.} \quad (3.4)$$

The standard Kato theory (Theorem 2.2 here) gives a smoothing estimate for the Schrödinger flow:

$$\| |x|^{-1} e^{it(-\Delta-V)} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2}. \quad (3.5)$$

Then the full set of Strichartz estimates follows from (3.5), via the usual Rodnianski-Schlag trick [21].

In order to apply the same procedure to the wave equation, in [2] the following estimate for the wave flow is proved:

$$\| |x|^{-1} e^{it\sqrt{-\Delta-V}} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}. \quad (3.6)$$

The proof in [2] is rather involved (see also the Errata relative to that paper); however, using the theory developed in Section 2, (3.6) follows directly by the combination of (3.4) and (2.10).



Note that we can prove additional estimates which are apparently new. For instance, we deduce the following homogeneous estimate for the Klein-Gordon flow

$$\| |x|^{-1} e^{it\sqrt{1-\Delta-V}} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}, \quad (3.7)$$

and nonhomogenous estimates for all the flows, like

$$\| |x|^{-1} \int_0^t e^{-i(t-s)(-\Delta+V)} F(s) ds \|_{L^2(\mathbb{R}^{n+1})} \lesssim \| |x| F \|_{L^2(\mathbb{R}^{n+1})}$$

for the Schrödinger equation and similar ones for the wave and Klein-Gordon equations.

**3.3. Wave equation with large magnetic potentials.** We conclude the paper by proving Strichartz estimates for a wave equation on  $\mathbb{R}^n$ ,  $n \geq 3$ , of the form

$$u_{tt} + Hu = 0$$

where  $H$  is a magnetic Schrödinger operator

$$H = (i\nabla + A(x))^2 + V(x). \quad (3.8)$$

We make the following assumptions:  $A(x) \in \mathbb{R}^n$ ,  $V(x) \in \mathbb{R}$ , moreover for some  $C, \epsilon_0 > 0$

$$|A(x)| + \langle x \rangle |V(x)| \leq C \langle x \rangle^{-1-\epsilon_0}, \quad (3.9)$$

$$\forall 0 < \epsilon_1 < \epsilon_0, \quad \langle x \rangle^{1+\epsilon} A(x) \in \dot{H}_{2n}^{\frac{1}{2}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \quad (3.10)$$

where  $\dot{H}_q^{\frac{1}{2}}$  is the space with norm  $\| |D|^{\frac{1}{2}} u \|_{L^q}$ ,  $|D| = (-\Delta)^{\frac{1}{2}}$ ; finally, we assume that 0 is not an eigenvalue of  $H$  nor a resonance, in the sense that

$$\langle x \rangle^{\frac{n-4}{2}-} u(x) \in L^2(\mathbb{R}^n), \quad Hu = 0 \implies u \equiv 0. \quad (3.11)$$

Clearly in dimension  $n \geq 5$  it is sufficient to assume that 0 is not an eigenvalue. Then the results in [9] imply the following resolvent estimate:

**Theorem 3.1.** *Assume the operator  $H$  in (3.8) is selfadjoint, nonnegative, and satisfies (3.9), (3.10) and (3.11). Then the resolvent  $R(z) = (H - z)^{-1}$  satisfies for  $\delta > 0$*

$$\| \langle x \rangle^{-\frac{1}{2}-\delta} |D|^{\frac{1}{2}} R(z) |D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\delta} f \|_{L^2} + \| \langle x \rangle^{-1-\delta} R(z) \langle x \rangle^{-1-\delta} f \|_{L^2} \leq C_\delta \|f\|_{L^2} \quad (3.12)$$

with a constant uniform in  $z \in \mathbb{C} \setminus \mathbb{R}$ .

*Proof.* The estimate follows from Theorem 1.2 in [9] with the choice  $\alpha = 1/2$ . The result is stated there in the form of a limit absorption principle i.e., for  $z = \lambda^2 + i0$ ; the form given here follows from the remark that the spectrum is positive by assumption, and then by a simple application of the Phragmén-Lindelöf principle on the upper resp. lower complex plane.  $\square$

In the terminology of Kato-Yajima, Theorem 3.1 states that  $\langle x \rangle^{-\frac{1}{2}-\delta} |D|^{\frac{1}{2}}$  and  $\langle x \rangle^{-1-\delta}$  are  $H$ -supersmoothing operators. Applying Theorem 2.4 we get the estimate

$$\| \langle x \rangle^{-\frac{1}{2}-\delta} |D|^{\frac{1}{2}} e^{it\sqrt{H}} f \|_{L^2(\mathbb{R}^{n+1})} + \| \langle x \rangle^{-1-\delta} e^{it\sqrt{H}} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \| H^{\frac{1}{4}} f \|_{L^2(\mathbb{R}^n)} \quad (3.13)$$

for the corresponding wave flow. Then we can prove the full set of non-endpoint Strichartz estimates:

**Theorem 3.2.** *Let  $n \geq 3$ . Assume the operator  $H$  in (3.8) is selfadjoint, non-negative, and satisfies (3.9), (3.10) and (3.11). Then the wave flow satisfies the non-endpoint Strichartz estimates*

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} e^{it\sqrt{H}} f \|_{L^p L^q} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}, \quad (3.14)$$

and

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} \sin(t\sqrt{H}) H^{-\frac{1}{2}} g \|_{L^p L^q} \lesssim \|g\|_{\dot{H}^{-\frac{1}{2}}}, \quad (3.15)$$

for all  $2 < p \leq \infty$ ,  $2 \leq q < \frac{2(n-1)}{(n-3)}$  with  $2p^{-1} + (n-1)q^{-1} = (n-1)2^{-1}$ .

*Remark 3.1.* It is possible to prove the endpoint estimate using the following non-homogeneous mixed Strichartz-smoothing estimate for the free wave flow ( $n \geq 4$ )

$$\| \int_0^t e^{i(t-s)|D|} F(s) \|_{L^2 L^{\frac{2(n-1)}{n-3}}} \lesssim \| \langle x \rangle^{\frac{1}{2}+} |D|^{\frac{1}{2}} F \|_{L^2 L^2}. \quad (3.16)$$

This estimate can be obtained by a modification of the techniques used in [14] for the corresponding result for the Schrödinger equation. We prefer to omit the details here.

*Proof of Theorem 3.2.* The function  $u = e^{it\sqrt{H}} f$  solves the Cauchy problem

$$u_{tt} - \Delta u = -i\nabla \cdot (Au) - iA \cdot \nabla u - (|A|^2 + V)u, \quad u(0, x) = f, \quad u_t(0, x) = i\sqrt{H}f$$

hence we can write

$$e^{it\sqrt{H}} f = \cos(t|D|)f + i \sin(t|D|)|D|^{-1}\sqrt{H}f - i\widetilde{I} - i\widetilde{II} - \widetilde{III} \quad (3.17)$$

where

$$\widetilde{I} = \int_0^t \sin((t-s)|D|)|D|^{-1}\nabla \cdot (Au)ds, \quad \widetilde{II} = \int_0^t \sin((t-s)|D|)|D|^{-1}A \cdot \nabla uds$$

and

$$\widetilde{III} = \int_0^t \sin((t-s)|D|)|D|^{-1}(|A|^2 + V)uds.$$

In the following we shall estimate as usual the more general operators

$$I = \int_0^t e^{i(t-s)|D|}|D|^{-1}\nabla \cdot (Au)ds, \quad II = \int_0^t e^{i(t-s)|D|}|D|^{-1}A \cdot \nabla uds$$

and

$$III = \int_0^t e^{i(t-s)|D|}|D|^{-1}(|A|^2 + V)uds$$

since the estimates for  $I, II, III$  imply the corresponding estimates for  $\widetilde{I}, \widetilde{II}, \widetilde{III}$ .

Besides the fundamental smoothing estimates (3.13), we need the following tools: the homogeneous Strichartz estimates for the free wave equation

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} e^{it|D|} f \|_{L^p L^q} \lesssim \| |D|^{\frac{1}{2}} f \|_{L^2}, \quad (3.18)$$

the dual smoothing estimate for the free wave equation

$$\| \int e^{-is|D|} F(s)ds \|_{L^2} \lesssim \| \langle x \rangle^{\frac{1}{2}+} F \|_{L^2 L^2}, \quad (3.19)$$

and the fact that the operators

$$\langle x \rangle^{\frac{1}{2}+\epsilon} |D|^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon'} |D|^{\frac{1}{2}} \quad \text{and} \quad \langle x \rangle^{\frac{1}{2}+\epsilon} |D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon'} |D|^{-\frac{1}{2}} \quad (3.20)$$

are bounded on  $L^2$  provided  $0 < \epsilon < \epsilon'$  are small enough. Estimates (3.18) are well known; (3.19) is the dual of

$$\| \langle x \rangle^{-\frac{1}{2}-\epsilon} e^{it|D|} f \|_{L^2 L^2} \lesssim \| f \|_{L^2}$$

which follows from Theorem 2.4 and the  $-\Delta$ -smoothness of  $\langle x \rangle^{-\frac{1}{2}-\epsilon} |D|^{\frac{1}{2}}$  proved in Section 3.1. Finally, the  $L^2$  boundedness of (3.20) can be proved by interpolation, or a direct proof can be found in Lemma 6.2 in [9].

Note also that by Hardy's inequality, since  $|A| \lesssim |x|^{-1}$  and  $|V| \lesssim |x|^{-2}$ , we have

$$\| H^{\frac{1}{2}} f \|_{L^2}^2 \leq \| \nabla u + iAu \|_{L^2}^2 + \| |V|^{\frac{1}{2}} u \|_{L^2}^2 \lesssim \| f \|_{H^1}^2.$$

Thus by interpolation we obtain

$$\| H^{\frac{1}{4}} f \|_{L^2} \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}}. \quad (3.21)$$

Equivalently, the operator  $H^{\frac{1}{4}} |D|^{-\frac{1}{2}}$  is bounded on  $L^2$  and hence also the operator

$$|D|^{-\frac{1}{2}} H^{\frac{1}{2}} |D|^{-\frac{1}{2}} = (H^{\frac{1}{4}} |D|^{-\frac{1}{2}})^* H^{\frac{1}{4}} |D|^{-\frac{1}{2}} \quad (3.22)$$

is  $L^2$  bounded.

The estimate of the homogenous terms in (3.17) follows directly from (3.18); in particular, the second term can be estimated by

$$\lesssim \| |D|^{\frac{1}{2}} |D|^{-1} \sqrt{H} f \|_{L^2} = \| |D|^{-\frac{1}{2}} H^{\frac{1}{2}} |D|^{-\frac{1}{2}} |D|^{\frac{1}{2}} f \|_{L^2} \lesssim \| |D|^{\frac{1}{2}} f \|_{L^2}$$

using the boundedness of (3.22).

We now focus on the main terms  $I, II, III$ . By the usual application of Christ-Kiselev's Lemma, in the non-endpoint case it is sufficient to estimate the untruncated integrals which can be split as

$$e^{it|D|} \int e^{-is|D|} F(s) ds.$$

Using first the homogeneous Strichartz estimate (3.18) to bound  $e^{it|D|}$ , then the dual smoothing estimate (3.19), we obtain

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} \int e^{i(t-s)|D|} F(s) ds \|_{L^p L^q} \lesssim \| \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{\frac{1}{2}} F \|_{L^2 L^2}.$$

For the term  $I$  this gives

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} I \|_{L^p L^q} \lesssim \| \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{-\frac{1}{2}} \nabla \cdot (Au) \|_{L^2 L^2}.$$

By the boundedness of the Riesz operator  $\nabla |D|^{-1}$  in weighted  $L^2$  spaces (note that  $\langle x \rangle^{\frac{1}{2} + \epsilon}$  is an  $A_2$  weight) we get

$$\lesssim \| \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{\frac{1}{2}} (Au) \|_{L^2 L^2}.$$

Writing for  $\epsilon' > \epsilon$

$$\langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{\frac{1}{2}} = \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon'} |D|^{-\frac{1}{2}} \cdot |D|^{\frac{1}{2}} \langle x \rangle^{\frac{1}{2} + \epsilon'}$$

and recalling (3.20), we arrive at

$$\lesssim \| |D|^{\frac{1}{2}} (\langle x \rangle^{\frac{1}{2} + \epsilon'} Au) \|_{L^2 L^2} = \| |D|^{\frac{1}{2}} (\langle x \rangle^{1+2\epsilon'} A \cdot \langle x \rangle^{-\frac{1}{2} - \epsilon'} u) \|_{L^2 L^2}.$$

The Kato-Ponce inequality gives

$$\begin{aligned} & \| |D|^{\frac{1}{2}} (\langle x \rangle^{1+2\epsilon'} A \cdot \langle x \rangle^{-\frac{1}{2} - \epsilon'} u) \|_{L^2} \lesssim \\ & \lesssim \| \langle x \rangle^{1+2\epsilon'} A \|_{\dot{H}_{2n}^{\frac{1}{2}}} \| \langle x \rangle^{-\frac{1}{2} - \epsilon'} u \|_{L^{\frac{2n}{n-1}}} + \| \langle x \rangle^{1+2\epsilon'} A \|_{L^\infty} \| |D|^{\frac{1}{2}} (\langle x \rangle^{-\frac{1}{2} - \epsilon'} u) \|_{L^2} \end{aligned}$$

and by the Sobolev embedding  $\dot{H}^{\frac{1}{2}} \subset L^{\frac{2n}{n-1}}$  we obtain

$$\lesssim C_A \cdot \| |D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon'} u \|_{L^2}$$

where we used the assumptions on  $A$ , provided  $\epsilon, \epsilon'$  are small enough. Finally, writing

$$|D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon'} = |D|^{\frac{1}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon'} |D|^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2} + \epsilon''} \cdot \langle x \rangle^{-\frac{1}{2} - \epsilon''} |D|^{\frac{1}{2}}$$

and using again the boundedness of (3.20) and the smoothing estimate (3.13), we get

$$\lesssim C_A \cdot \| \langle x \rangle^{-\frac{1}{2} - \epsilon''} |D|^{\frac{1}{2}} u \|_{L^2 L^2} \lesssim \| H^{\frac{1}{4}} f \|_{L^2} \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}}.$$

For the term  $II$ , proceeding as before we have

$$\| |D|^{\frac{1}{q} - \frac{1}{p}} II \|_{L^p L^q} \lesssim \| \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{-\frac{1}{2}} (A \cdot \nabla u) \|_{L^2 L^2}.$$

We use the decomposition

$$\langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{-\frac{1}{2}} A \cdot \nabla = T_1 S T_2 \cdot \langle x \rangle^{-\frac{1}{2} - \epsilon''} |D|^{\frac{1}{2}}$$

where, for some  $0 < \epsilon < \epsilon'' < \epsilon'$ ,

$$T_1 = \langle x \rangle^{\frac{1}{2} + \epsilon} |D|^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{2} - \epsilon'} |D|^{\frac{1}{2}},$$

$$S = |D|^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2} + \epsilon'} A \langle x \rangle^{\frac{1}{2} + \epsilon'} |D|^{\frac{1}{2}} \equiv |D|^{-\frac{1}{2}} \langle x \rangle^{1 + \epsilon' + \epsilon''} A |D|^{\frac{1}{2}}$$

and

$$T_2 = |D|^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon'} \nabla |D|^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}+\epsilon''}.$$

The operator  $T_1$  is  $L^2$  bounded by (3.20); this is true also for  $T_2$  since

$$(T_2)^* = \langle x \rangle^{\frac{1}{2}+\epsilon''} \nabla |D|^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon'} |D|^{-\frac{1}{2}}$$

and we can use the boundedness of the Riesz operator  $\nabla |D|^{-1}$  in weighted  $L^2$  and then again (3.20). Finally, the operator  $S$  or rather  $S^*$  is  $L^2$  bounded again by the Kato-Ponce inequality ( $\delta = \epsilon' + \epsilon''$ )

$$\| |D|^{\frac{1}{2}} \langle x \rangle^{1+\delta} A |D|^{-\frac{1}{2}} f \|_{L^2} \lesssim \| \langle x \rangle^{1+\delta} A \|_{\dot{H}^{\frac{1}{2n}}_{2n}} \| |D|^{-\frac{1}{2}} f \|_{L^{\frac{2n}{n-1}}} + \| \langle x \rangle^{1+\delta} A \|_{L^\infty} \| f \|_{L^2}$$

and the Sobolev embedding  $\dot{H}^{\frac{1}{2}} \subset L^{\frac{2n}{n-1}}$ . In conclusion we get as above

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} II \|_{L^p L^q} \lesssim C_A \cdot \| \langle x \rangle^{-\frac{1}{2}-\epsilon''} |D|^{\frac{1}{2}} u \|_{L^2 L^2} \lesssim \| H^{\frac{1}{4}} f \|_{L^2} \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}}.$$

For the last term  $III$  the computations are similar: we have

$$\| |D|^{\frac{1}{q}-\frac{1}{p}} III \|_{L^p L^q} \lesssim \| \langle x \rangle^{\frac{1}{2}+\epsilon} |D|^{-\frac{1}{2}} (Vu) \|_{L^2 L^2},$$

then by the  $L^2$  boundedness of

$$\langle x \rangle^{\frac{1}{2}+\epsilon} |D|^{-\frac{1}{2}} \langle x \rangle^{-\frac{1}{2}-\epsilon'} |D|^{\frac{1}{2}}$$

for  $\epsilon' > \epsilon$  we arrive at

$$\lesssim \| |D|^{-\frac{1}{2}} \langle x \rangle^{\frac{1}{2}+\epsilon'} Vu \|_{L^2 L^2}.$$

Next we use Sobolev embedding, the assumption  $|V| \lesssim \langle x \rangle^{-1-\epsilon_0}$  and Hölder's inequality

$$\lesssim \| \langle x \rangle^{\frac{1}{2}+\epsilon'} Vu \|_{L^2 L^{\frac{2n}{n-1}}} \lesssim \| \langle x \rangle^{-\frac{1}{2}-\epsilon''} \|_{L^{2n}} \| \langle x \rangle^{-1-\epsilon'''} u \|_{L^2 L^2}$$

with  $\epsilon' + \epsilon'' + \epsilon''' = \epsilon_0$ , and recalling (3.13) we conclude the proof of the first estimate in (3.14).

The proof of the second estimate (3.14) is identical: just notice that the function  $u = \sin(t\sqrt{H})H^{-\frac{1}{2}}g$  solves the wave equation

$$u_{tt} + Hu = 0, \quad u(0, x) = 0, \quad u_t(0, x) = g$$

and proceed as above.  $\square$

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